Structuring Co-Constructive Logic for Proofs and Refutations

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Abstract. This paper considers a topos-theoretic structure for the interpretation of co-constructive logic for proofs and refutations following [49]. It is notoriously tricky to define a proof-theoretic semantics for logics that adequately represent constructivity over proofs and refutations. By developing abstractions of elementary topoi, we consider an elementary topos as structure for proofs, and complement topos as structure for refutation. In doing so, it is possible to consider a dialogue structure between these topoi, and also control their relation such that classical logic (interpreted in a Boolean topos) is simulated where proofs and refutations are conclusive.

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1. Introduction

1.1. Constructivism beyond intuitionism

In [49], it was argued that properties such as excluded middle and disjunction property (and even a constructive account of truth), do not capture what is central to a constructive logic. Rather, these properties are typical features of intuitionistic logic, whose constructivity lies in an ability to understand proofs epistemically requiring the existence of actual constructions, and where a proposition determined by its proofs.[1] By taking this as the central feature of constructive logic, it is possible to consider logics that may be constructive whilst not sharing properties typical to intuitionism. More specifically, we will investigate the dual of intuitionistic logic, co-intuitionistic

[1]This is also taken to rule out the "potentialist" interpretation of proof [34] (and at certain points held by Dummett [11]), in favor of an actualist, time-bound, notion of proof [18]. For discussion, see [30, 36, 40].

logic, as a logic of refutation, alongside intuitionistic logic of proofs [49]. Direct proof and refutation are dual to each other, and are constructive, whilst there also exist syntactic, weak, negations within both logics. In this respect, the logic of refutation is weakly paraconsistent in the sense that it allows for statements for which, neither they, nor their negation, are refuted. In brief, this means that, whilst intuitionistic logic L_I allows us to deal with undetermined formulas:
Undetermined: $\vdash\ \alpha \vee \neg\alpha$ is not a theorem;
co-intuitionistic logic for refutation, L_C, allows us to deal with undetermined formulas also, since the dual of "Undetermined" holds:
Inconsistent: $\dashv\ \alpha \wedge \neg\alpha$ is not a "counter-theorem" (where $\dashv$ is a "refutation" relation).
The latter should be understood constructively, as saying that there may exists some formula α, for which neither α, nor its negation are refuted.

Taken together, the logics constitute a co-constructive logic for proofs and refutations, which is intuitive in the context of a re-interpretation of Kolmogorov's logic of problems [22].[2] There, the suggestion is that constructive logic has to do "not with theoretical propositions but, on the contrary, problems".[3] Following Kolmogorov, we can distinguish between problems (which concern reasoning and argument), and solutions, which are propositions (which concern truth and falsity).[4] These will be interpreted in the below as proof and refutation *attempts* and conclusively valid proofs and refutations, respectively. Kolmogorov understands refutations as reduction to contradiction, but, instead, we will approach the notion directly here, on a par with proof. Taken together, the situation is as follows. For any statement, we consider it as a declarative statement in the form of a question: γ? This may be considered by prover and refuter as a tentative hypothesis, to be "tested". If we think of these in terms of proof and refutation attempts, then the following Brouwer-Heyting-Kolmogorov (BHK) style clauses capture the relationship between proof and refutation attempts for conjunction and disjunction:

- c is a proof attempt of $\alpha \wedge \beta$ if c is a pair (c_1, c_2) such that c_1 is a proof attempt of α and c_2 is a proof attempt of β.

[2] As pointed out by an anonymous reviewer, it may be more appropriate to call this *bi*-, rather than *co*-constructive logic. I agree, but stick with the original formulation, since, whilst it has become mathematical practice to use the prefix "co" to refer to a dual (rather than the pair), the etymology of the prefix as having to do with joint partnership and mutual practice in English captures their relationship somewhat better.

[3] 'In addition to theoretical logic, which systematizes a proof schemata for theoretical truths, one can systematize a proof schemata for solutions to problems [...] In the second section, assuming the basic intuitionistic principles, intuitionistic logic is subjected to a critical study; it is thus shown that it must be replaced by the calculus of problems, since its objects are in reality problems, rather than theoretical propositions' [22, p.58] (Translated in [27]).

[4] This is reminiscent of Martin-Löf's [29] separation of judgments and propositions: 'A proof is, not an object, but an act [. . .], and the act is primarily the act as it is being performed, only secondarily, and irrevocably, does it become the act that has been performed.'

- c is a refutation attempt of $\alpha \wedge \beta$ if c is a pair (l, c') such that c' is a refutation attempt of α or c is a pair (r, c') such that c' is a refutation attempt of β.
- c is a proof attempt of $\alpha \vee \beta$ if c is a pair (l, c') such that c' is a proof attempt of α or c is a pair (r, c') such that c' is a proof attempt of β.
- c is a refutation attempt of $\alpha \vee \beta$ if c is a pair (c_1, c_2) such that c_1 is a refutation attempt of α and c_2 is a refutation attempt of β.

Of course, neither of the refutation-attempt clauses are acceptable intuitionistically (where refutations can not be treated symmetric with proofs), or classically, where refutations are typically dealt with by appeal to counter-models, and in any case, neither the disjunction property (for proofs) nor conjunction property (for refutations) holds. These are required for any constructive approach to inquiry, since we should not be able to say that some formula is proved, without also being able to provide a proof of any of its premises. In classical logic, this fails by contraction, so there may exist a disjunction $\alpha \vee \beta$ without any means of deciding which formula is proved. An additional complication is that, if we want to put proofs and refutations on the same level (and presumably non-interdefinable), then we will need to allow for cases in which we have a "proof" of some α, whilst also allowing for the possibility of searching to see whether or not it can be refuted (i.e. if a "refutation" of α can be found).

This makes way for an internal duality between proofs and refutations (in L_I and L_C, respectively), which renders the logic of problems symmetrical to instantiate "dialogue" between "prover" and "refuter".[5]

1.2. Proof-theoretic semantics

This paper primarily examines the construction of a proof-theoretic semantics for this co-constructive logic. In general, it is desirable in constructing such semantics that we should be capable of explicitating proofs, rather than just establishing whether or not statements are true or false. So, the development of proof-theoretic semantics attempts to characterize logical consequence (and reasoning, more generally), in terms of proof, such that the way in which we construct a proof is not lost.

An immediate difficulty for constructing proof-theoretic semantics is due to an inability of intuitionistic logic to adequately deal with refutation and falsity as pointe out in [40, 39]. The primacy of proof (in contrast to refutation) is problematic for the development of a semantics of proofs, where we do not want to rely upon a counter-model as stand-in for refutation. As brief (and rough) example, take Γ^+ as a set of sentential theorems, V as some model (ordinarily set-theoretic), with $\vdash$ a derivability relation, and $\models$

[5]The following is indicative of the approach that we take: 'This permutation Prover $\leftrightarrow$ Denier should be understood as an internal symmetry of logic itself. An analogy with Galilean Mechanics emerges here, which we find useful to mention since it guided our work. The idea is that the symmetry Prover $\leftrightarrow$ Denier plays the same role as a uniform and rectilinear change of frame of reference in Galilean Mechanics' [31].

a model-satisfaction relation. Then, in order to construct a proof-theoretic semantics, we would like to (at the least) move from requiring that, for every formula α:

either $\exists\Gamma^+(\Gamma^+ \vdash \alpha)$ or, $\exists V(V \not\models \alpha)$

to:

either $\exists\Gamma^+(\Gamma^+ \vdash \alpha)$ or, $\exists\Delta^-(\Delta^- \dashv \alpha)$ (where Δ^- is a set of sentential counter-theorems, or refutations).

However, the asymmetry of proof and refutation in intuitionistic logics renders this impossible. This is reflected by the restriction on intuitionistic sequents to at most a single formula on the *r.h.s* of the turnstile;[6] the indirect interpretation of refutation as reduction to absurdity; and the fact that a proof of α is inequivalent to $\neg\neg\alpha$.[7] Of course, this is complicated by the fact that it is these restrictions that ensure constructivity, for example, ensuring by the disjunction property, that a (cut-free) proof of $\alpha \vee \beta$ (where both α and β have single, cut-free proofs), must be a proof of one of the disjuncts.[8] In logics without this restriction, such as classical logic, this fails by contraction, so there may exist a disjunction $\alpha \vee \beta$ without any means of deciding which formula is proved, which is problematic if we pursue a semantics of proof.[9]

1.3. Bi-intuitionism

In order to deal with the above issues, we are considering a co-constructive logic comprising intuitionistic logic for proofs, and co-intuitionistic logic for refutations.[10] This builds upon analysis of the relationship between intuitionistic and co-intuitionistic logic as discussed in [4, 5, 7, 12, 16, 35, 37, 39, 52]; and systems of proof and refutation discussed in [41, 42, 43, 44, 45, 53, 54]. Unlike the construction suggested here, the typical relationship between intuitionistic and co-intuitionistic logic in the literature is not without significant

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[6]Note that this feature is not forced on intuitionistic sequents, though restrictions on rules or dependency relations are required to ensure that only intuitionistic derivations are valid in sequents with multiple conclusions, see [9] for discussion. There are, nonetheless, other reasons to think that multiple-conclusion sequents are inappropriate for the development of a proof-theoretic semantics in a constructive, inferentialist tradition, see the discussion in [47, 50].

[7]Furthermore, as Shramko et al. [40] point out, if we define falsity in terms of negation, then we are led to a reliance, not only on a syntactic feature (negation), but also on truth, and, as Dummett [10] is aware, this, leads to bivalence under commonly-held assumptions regarding the nature of proofs [see author reference omitted for discussion].

[8]To get ahead of ourselves, this property is mirrored in co-intuitionistic logic by the "conjunction property", where $\dashv \alpha \wedge \beta$ iff $\dashv \alpha$ or $\dashv \beta$.

[9]Lafont [14, Appendix B.1] argues that 'classical logic is inconsistent, not from a logical viewpoint (false is not provable), but from an algorithmic one', since proofs can not be considered algorithmically, and so 'classical logic has no denotational semantics, except the trivial one which identifies all the proofs of the same type'. It is worth noting that, as an anonymous reviewer suggests, this result is limited to standard sequent calculi for classical logic, since there do exist algorithmic interpretations using the Curry-Howard-Lambek correspondence and type theories for classical logic.

[10]This is dissimilar to constructions along the lines suggested by Nelson [33] since they typically assume that falsity is equivalent to a negated truth.

issues, and often the relationship between the two is left without intuitive interpretation.[11] For example, there have been a number of attempts to allow the two logics to "cohabit" in the same structure have been explored by means of bi-intuitionistic calculi. For example, [37] develops the algebraic analysis of co-intuitionism given in [16], by means of bi-Heyting algebra, which combines a Heyting algebra with the dual, co-Heyting (or Brouwerian), algebra. This is equivalent, on Rauszer's [37] approach, to an extension of intuitionistic logic with a co-implication operator, which is sometimes called subtraction. Just as negation can be defined by means of implication and 0 in a Heyting algebra, so negation can be defined by means of co-implication and 1 in a co-Heyting algebra: $\neg_C \alpha =_{df} 1 \Leftarrow \alpha$, where $\Leftarrow$ is a co-implication operator (roughly, this is read "1 without α".[12]

However, this construction of bi-intuitionism runs into significant difficulties regarding the construction of an adequate proof-theory, and proof-theoretic semantics. In particular, it is difficult to ensure that bi-intuitionism does not simply collapse to a (semi-)classical single logic due to the simulation of an involution within the combined structure.[13] The issue is clarified on consideration of the possibility of a categorical interpretation of bi-intuitionism. It is well-known that constructive logic suspends the principle of excluded middle as theorem. Of course, this does not mean that principle of excluded middle is always rejected, rather its use is controlled by context. This can be studied structurally by means of a cartesian closed category (CCC), which can adequately characterize the notion of constructive proof.[14] However, it is shown in [7], that, for any CCC (with final and initial objects), if it is extended by co-exponentials (which adequately represent the co-implication operator), then it collapses to a single partial order, and so fails to provide adequate proof-theoretic semantics.

1.4. Symmetric dialogue topoi

In the context of co-constructive logic, the aforementioned issues relating to "collapse" are rendered more perspicuous, such that the mechanism by which collapse occurs can be exploited and controlled. This is possible by constructing a proof-theoretic semantics by means of a topos-theoretic construction. It is well-known that the internal logic of an elementary topos is useful as a model of constructive provability. For example, in many topoi, axiom of choice fails, law of excluded middle does not hold unrestrictedly, and the double negation of a formula does not imply that formula. Resultantly, they provide a natural environment for understanding intuitionistic

[11]With exception to the latter: [39] explores co-intuitionistic logic as Popperian logic, and [4] develops a polarized proof-theory for the logic of pragmatics as extension of classical logic.

[12]Classically, a co-implication "$\beta \Leftarrow \alpha$" is defined as "$\beta \wedge \neg\alpha$".

[13]For discussion of this issue, see below, and [4, 7]. Note that, as both authors point out, this is not the case for bi-intuitionistic linear logic, which does not collapse when merging both intuitionistic and co-intuitionistic logic, [3].

[14]See, for example, [23].

logic. Furthermore, in [32] (and developed in [13]), it is shown that it is possible to construct complement-topoi whose internal logic is paraconsistent, so providing a natural environment for understanding co-intuitionistic logic.

Our analysis of complement topoi builds upon that work. However, unlike those constructions, we do not attempt to a build these as full mathematical structures with internal, paraconsistent, logic. Instead, we construct two, dual, topoi representing a symmetric dialogue structure for proofs and refutations. This is made possible by syntactically decoupling the two logics, by which we gain constructivity over proofs and refutations, and traction on "weak" negations expressing the absence of proof (or refutation).[15] Then, by identifying the conditions under which they collapse to a Boolean topos, we can allow for collapse under controlled conditions, which are just those in which we have conclusive proofs or refutations (as solutions). Such a system is capable of non-trivially dealing with the simultaneous consideration of proof and refutation *attempts* of the same formula, so retaining constructivism over proofs and refutations.

1.5. Outline

We begin by introducing in §2 an abstract definition of logic and duality, before providing algebraic structures for Heyting and co-Heyting algebra. As we show, these are adequate structures for understanding intuitionist and co-intuitionist logics, with a duality mapping between the two. Then, §3 briefly provides an overview of how the two logics are understood as co-constructive over proof and refutation attempts. In §4, a basic categorical construction for intuitionistic logic is provided, before going on to show that it collapses in the presence of co-intuitionistic co-implication. Then, §5 provides an account of dual topoi as dialogue structure over proofs and refutations. We note several features of these structures in relation to the constructivity of the internal logics, before introducing a specific relationship between them, called coherence. Coherence accounts for the conditions under which they collapse to a Boolean topos. §6 finishes by discussing the three resultant topos-theoretic structures and proof-theoretic semantics.

2. Logic and algebraic duality

First, we provide a generalised and suitably abstract definition of a "logic", which we will think of as a structure of entailment, since we are primarily interested in a proof-theoretic account.[16]

Definition 2.1. Say that a "logic", L, is an entailment structure, which is an ordered pair, $(S, \vdash_L)$, where S is a denumerable set of propositional formulas, and $\vdash_L$ is a binary entailment relation defined on $P(S) \times P(S)$ ($P(S)$ is the set of all finite subsets of S), which we call normal when $\vdash_L$ is reflexive, transitive, monotonic and finitary.

[15]These are similar to the familiar concept of "negation as failure".

[16]See [48] for a similar definition.

An entailment structure defined this way is just a just a pre-order where the ordering relation $\leq$ on S is equivalent to $\vdash_L$ (i.e. $\alpha \leq \beta \Leftrightarrow \alpha \vdash_L \beta$).[17] Entailment structures can be restricted in different ways, which we shall think of in terms of sequents in a logic.

Definition 2.2. A sequent is just an ordered pair, $\langle \Gamma, \Delta \rangle$ where Γ, Δ are finite (possibly empty) sequences of formulas of S. Say that a right-asymmetric sequent (Γ, α) is restricted to at most a single formula on the right; a left-asymmetric sequent (α, Γ) is restricted to at most a single formula on the left, and a symmetric sequent (Γ, Δ) has no such restrictions. A sequent rule $\mathcal{R}$ in any logic L is an ordered pair consisting of a finite sequence of sequent premises and a sequent conclusion $\mathcal{R} = \langle \{SEQ^P\}, SEQ^C \rangle$, and, in case the list of premises is empty, the instance of a rule is called an axiom.

In this way, we may think of a specific logic to be determined by a proof structure (set of axioms and sequent rules), where any collection of sequents $\mathcal{S}$ that is closed under standard structural rules determines a finitary, normal, logic. For example, it is the case that $\Gamma \vdash \alpha$ iff for some finite $\Gamma_0 \subseteq \Gamma$, we have $(\Gamma_0 \vdash \alpha) \in \mathcal{S}$.[18]

Now, consider the construction of formulas in the languages for the two logics that we are primarily interested in here, intuitionistic logic L_I, which I take to deal with proofs, and L_C for co-intuitionistic logic, which I take to deal with refutations.

Definition 2.3. Define two languages S, S^d over a denumerable set of atomic formulas, for L_I and L_C respectively, in Backus-Naur form (α^+ is any atomic formula of S, α^- is any atomic formula of S^d):[19]

(S) $\beta^+ ::= \alpha^+ | \neg_I \beta^+ | \beta^+ \wedge \beta^+ | \beta^+ \vee \beta^+ | \beta^+ \Rightarrow \beta^+ | 0^+$

(S^d) $\beta^- ::= \alpha^- | \neg_C \beta^- | \beta^- \wedge \beta^- | \beta^- \vee \beta^- | \beta^- \Leftarrow \beta^- | 1^-$

Here, $\neg_I$ and $\neg_C$ denote the negations of the two languages, and $\Rightarrow$ and $\Leftarrow$ denote implication and co-implication, respectively. These are the key distinctions with classical logic, though as I show below, $\Leftarrow$ essentially operates as a kind of "implication for refutations".[20] Note also that we have made a

[17]The definition is necessarily abstract, there will be many further constraints on the structure in any specific logic. It is also worth noting that $(S, \leq)$ will be a poset in case $\leq$ is also anti-symmetric, which is the case if we take the skeleton of the pre-order and the construct $\leq$ from the set of equivalence classes of S under logical equivalence ($\alpha \equiv \beta \leftrightarrow \alpha \leq \beta$ and $\beta \leq \alpha$). This may capture the idea that in propositional logic we may say that non-identical formulas e.g. α, $\alpha \vee \alpha$ have the same "inferential force" [p.158]Restall2000-RESAIT.

[18]For symmetric sequents, this is $\Gamma \vdash\dashv \Delta$ iff for some finite $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ we have $(\Gamma_0 \vdash\dashv \Delta_0) \in \mathcal{S}$. See also [20, p.113]. Note that I use $\vdash\dashv$ rather than typical $\vdash$ for symmetric sequents to highlight that they can be read in both directions.

[19]For the most part, I shall drop the superscripts where they are obvious from context.

[20]Though see [16, 37, 52] for alternative interpretations of $\Leftarrow$. Note also that both $\neg_C$ and $\Leftarrow$ can, however, be defined inside classical logic, in which case the duality mapping given below can be easily transformed into a full self-duality for classical logic, with $\Leftarrow$ dual to $\Rightarrow$, and $\neg_I$ and $\neg_C$ collapsing together. We should also note that an alternative usage of $\Leftarrow$ for a second implication exists in [6].

syntactic distinction between atoms of the two languages, to signify whether they are part of a proof or a refutation, and to control their interaction.

It is well known that the posetal structure of any intuitionist propositional logic is equivalent to a Heyting algebra.

Definition 2.4. (Heyting algebra) Let $(H, \leq)$ be a partial order with the following properties:[21]

Minimal element 0, such that, for all α, $0 \leq \alpha$;
Maximal element 1, such that, for all α, $\alpha \leq 1$;
$\alpha \wedge \beta \leq \alpha$;
$\alpha \leq \alpha \vee \beta$;
$(\alpha \wedge \beta \leq \gamma)$ iff $(\alpha \leq \beta \Rightarrow \gamma)$
$\neg\alpha = (\alpha \Rightarrow 0)$.

The relationship between a Heyting algebra and intuitionist propositional logic, L_I (with derivability relation denoted $\vdash_I$), is as follows.[22] Elements of $(H, \leq)$ are formulas of S; $\leq$ interprets $\vdash_I$; an intuitionistic sequent $\Gamma \vdash_I \alpha$ is $\beta_1 \wedge ... \wedge \beta_n \leq \alpha$ for all $\beta_i \in \Gamma$; and $\emptyset \vdash_I \alpha$ is interpreted as $1 \leq \alpha$; $\alpha \vee \beta = sup(\alpha, \beta)$; $\alpha \wedge \beta = inf(\alpha, \beta)$. By the definition of implication, we can rewrite the definition of negation as $\alpha \leq \neg\beta$ iff $\alpha \wedge \beta = 0$. It follows that $\alpha \wedge \neg\alpha = 0$, though it may not be the case that $\alpha \vee \neg\alpha = 1$, or $\alpha = \neg\neg\alpha$. It is a further property of implication that we have the familiar rule modus ponens so that $(\beta \Rightarrow \gamma) \wedge \beta \leq \gamma$. Then, the posetal structure of a Heyting algebra is equivalent to the logical poset for intuitionist logic, as defined by its typical sequent calculus LJ.

For any poset L, it is simple to define its dual.

Theorem 2.5. *If $L = (S, \leq)$ is a poset, then so is $L^d = (S, \leq^d)$, where $\leq^d$ is the converse of $\leq$ such that, for all $\alpha, \beta \in S$, $\beta \leq^d \alpha$ iff $\alpha \leq \beta$.*

Together with the definition of a logic, this gives us that: whenever $\alpha \leq_L \beta$, then $(\alpha, \beta) \in \vdash_L$; it follows that $(\beta, \alpha) \in \vdash_{L^d}$, so $\beta \leq_{L^d} \alpha$. This has particular significance in the context of constructing a dual algebra to Heyting algebra, which is commonly called co-Heyting algebra.[23]

Definition 2.6. (Co-Heyting algebra) Let $(C, \leq)$ be a partial order with the following properties [16]:

Minimal element 0, such that, for all α, $0 \leq \alpha$;
Maximal element 1, such that, for all α, $\alpha \leq 1$;
$\alpha \leq \alpha \wedge \beta$;
$\alpha \vee \beta \leq \beta$;
$(\alpha \leq \gamma \vee \beta)$ iff $(\alpha \Leftarrow \beta \leq \gamma)$
$\neg\alpha = 1 \Leftarrow \alpha$.

[21] The following is folklore, but further details can be found in [25, p. 50ff].
[22] What follows, and theorem 4 is well-known, for details see [51].
[23] Though sometimes it is called a Brouwerian algebra, for example, [16].

As mentioned above, we are considering the co-intuitionistic logic L_C (hence co-Heyting algebra) as a refutation structure, following the direction of [41, 44, 45].[24] A valid sequent "$\alpha \vdash \beta$" of a logic is ordinarily interpreted as saying that, under the assumption that there exists a proof of α, there exists a proof of β also. But, this interpretation is not forced upon us, and we should notice this is an argument in the form of a conditional.[25] Say, instead that we are inclined to assume that β is refuted, then "$\alpha \vdash \beta$" may be interpreted as saying that there exists a refutation of α, under that assumption.[26] Since intuitionistic logic is a "positive" logic that is interested in the providing of proofs, we can read an intuitionistic sequent in right-asymmetric form $\Gamma \vdash_I \alpha$ is that, whenever there is a proof of Γ, there is also a proof of α (in intuitionistic logic). The co-intuitionistic dual to a right-asymmetric sequent ($\Gamma \vdash \alpha$) is a left-asymmetric sequent ($\alpha \vdash \Gamma$) [52], but, we are interpreting co-intuitionism as a "negative" logic that is interested in the providing of refutations, so we are reading the latter as a refutation read from right to left.[27] As such, we shall reverse the turnstile so, that the meaning of a co-intuitionistic sequent of the form $\Gamma \dashv_C \alpha$ is that, under the assumption that Γ is refuted, α is refuted also. Just as $\emptyset \vdash_I \alpha$ indicates that α is a theorem of an intuitionistic proof-theory, $\emptyset \dashv_C \alpha$ indicates that α is a "counter-theorem" of a co-intuitionistic refutation-theory. It is important to note that the notion of refutation at work here should not definable by means of negation of a proof, since it is both symmetric, and on a par, with the latter, so, in general, $\alpha \dashv_C \beta \neq \neg_I \alpha \vdash_I \neg_I \beta$.

We can think of this in terms of two, dual, posets, with positive entailment, where $\alpha \vdash \beta$; and negative entailment, where $\beta \dashv \alpha$. In general, the former considers whether a proof of α entails a proof of β, and the latter, whether a refutation of β entails a refutation of α.

Definition 2.7. (Positive and negative entailment) Say that α positively entails β whenever $\alpha \vdash \beta$, so that $\alpha \leq_+ \beta$). Say that α negatively entails β whenever $\alpha \dashv \beta$, so that $\alpha \leq_- \beta$.

In the following, we consider co-intuitionistic logic by means of negative entailment, or refutation.

In this setting, the relationship between a co-Heyting algebra and co-intuitionistic propositional logic is as follows. Elements of $(C, \leq)$ are formulas of S^d; $\leq$ interprets $\dashv_C$; a sequent $\Gamma \dashv_C \alpha$ is $\alpha \leq \beta_1 \vee ... \vee \beta_n$ for all $\beta_i \in \Gamma$; and $\emptyset \dashv_I \alpha$ is interpreted as $\alpha \leq 0$; $\alpha \vee \beta = inf(\alpha, \beta)$; $\alpha \wedge \beta = sup(\alpha, \beta)$. By the

[24] Though note that refuted is here inequivalent with non-valid as Skura [41] has it.

[25] As pointed out in [38].

[26] Or if you prefer to think in terms of truth, then roughly, if $\alpha \vdash \beta$ is interpreted as saying that, if α is true, then β is true also, then $\alpha \dashv \beta$ is interpreted as saying that, if α is false, then β is false also.

[27] One reason for this interpretation is that it is "natural" in the sense that there is no decent implication operator in L_C, but there is a decent co-implication operator that strings together refuted formulas and obeys a dual-deduction theorem, as discussed in §5.

definition of co-implication, we can rewrite negation as $\neg\beta \leq \alpha$ iff $(\alpha \vee \beta) = 1$. So, $\alpha \vee \neg\alpha = 1$, but $\alpha \wedge \neg\alpha = 0$ need not hold, nor $\neg\neg\alpha = \alpha$. So, the posetal structure of a co-Heyting algebra is equivalent to the logical poset for co-intuitionist logic, L_C, as defined by the following proof-theoretic rules.

Definition 2.8. (Sequent calculus for co-intuitionistic logic of refutation [52, 49]):[28]

$$\alpha \dashv_C \alpha$$

$$\frac{\Delta \dashv_C}{\Delta \dashv_C \alpha}\ (\text{Weak-}R) \qquad \frac{\Delta \dashv_C \beta}{\Delta, \alpha \dashv_C \beta}\ (\text{Weak-}L)$$

$$\frac{\Delta, \alpha, \alpha \dashv_C \beta}{\Delta, \alpha \dashv_C \beta}\ (\text{Cont})$$

$$\frac{\Delta, \alpha, \sigma, \Theta \dashv_C \beta}{\Delta, \sigma, \alpha, \Theta \dashv_C \beta}\ (\text{Exchange}) \qquad \frac{\Delta, \alpha \dashv_C \beta \qquad \Gamma \dashv_C \alpha}{\Delta, \Gamma \dashv_C \beta}\ (\text{Cut})$$

$$\frac{\Delta \dashv_C \alpha}{\Delta \dashv_C \alpha \wedge \beta}\ (\wedge R_1) \qquad \frac{\Delta \dashv_C \beta}{\Delta \dashv_C \alpha \wedge \beta}\ (\wedge R_2)$$

$$\frac{\Delta, \alpha \dashv_C \sigma \qquad \Delta, \beta \dashv_C \sigma}{\Delta, \alpha \wedge \beta \dashv_C \sigma}\ (\wedge L)$$

$$\frac{\Delta \dashv_C \alpha \qquad \Delta \dashv_C \beta}{\Delta \dashv_C \alpha \vee \beta}\ (\vee R) \qquad \frac{\Delta, \alpha \dashv_C \sigma}{\Delta, \alpha \vee \beta \dashv_C \sigma}\ (\vee L_1)$$

$$\frac{\Delta, \beta \dashv_C \sigma}{\Delta, \alpha \vee \beta \dashv_C \sigma}\ (\vee L_2)$$

$$\frac{\Delta, \beta \dashv_C \alpha}{\Delta \dashv_C \alpha \Leftarrow \beta}\ (\Leftarrow R) \qquad \frac{\Delta, \alpha \dashv_C \sigma \qquad \Gamma \dashv_C \beta}{\Delta, \Gamma, \alpha \Leftarrow \beta \dashv_C \sigma}\ (\Leftarrow L)$$

$$\frac{\Delta, \alpha \dashv_C}{\Delta \dashv_C \neg_C \alpha}\ (\neg_C R) \qquad \frac{\Delta \dashv_C \alpha}{\Delta, \neg_C \alpha \dashv_C}\ (\neg_C L)$$

The duality relationship between the two structures is now much more simple to grasp. In general, the duality between Heyting and co-Heyting algebra is achieved by replacing any use of $\wedge, \vee, \Rightarrow, \neg, 1, 0$ in Heyting algebra, with

[28] I drop superscripts throughout.

$\vee, \wedge, \Leftarrow, \neg, 0, 1$ in co-Heyting algebra. More formally, we take the dualizing map $(-)^{\perp}$ between S and S^d as follows.

Definition 2.9. (Dualizing map $((-)^{\perp}) := L_I \mapsto L_C$) [e.g.][]Urbas1996-IGODL by induction on formulas):

$$\begin{gathered}
\alpha^{+\perp} := \alpha^{-} \text{ for atomic } \alpha \\
(\alpha \wedge \beta)^{\perp} := \alpha^{\perp} \vee \beta^{\perp} \\
(\alpha \vee \beta)^{\perp} := \alpha^{\perp} \wedge \beta^{\perp} \\
(\neg_I \alpha)^{\perp} := \neg_C (\alpha)^{\perp} \\
(\alpha \Rightarrow \beta)^{\perp} := \beta^{\perp} \Leftarrow \alpha^{\perp} \\
\{\alpha_1 ... \alpha_n\}^{\perp} := \{\alpha_1^{\perp} ... \alpha_n^{\perp}\}
\end{gathered}$$

We also stipulate that $(\alpha^{\perp})^{\perp} := \alpha$, so the mapping is easily extended to S^d.

3. Co-constructive logic

Before we examine categorical constructions for intuitionistic and co-intuitionistic logics, let us briefly consider how these are interpreted in co-constructive logic.[29]

It is possible to interpret L_I and L_C as separate structures in which, beginning with a sequent, rules are repeatedly applied until axioms are reached. However, this is not quite what we have in mind here. Rather, the idea is to think of the construction of a proof (refutation) as a process of argumentation that is characterised in a formal dialogue between the two entailment structures. In brief, we will consider a dialogue in the above structure in terms of a relationship between PROOF ATTEMPTS and REFUTATION ATTEMPTS that are "tests" of each other. Think of "prover" as making an attempt to provide a proof for some statement on the basis of a set of assumptions. We call this a PROOF-ATTEMPT, which informally is a sequence of arguments that is utilized by "prover" to suggest that, given a set of (assumed) premises, the final formula is provable. The success of a proof attempt, therefore, depends upon the existence of a proof for each assumption, and at each step of the argument. From the point of view of "refuter", the statement is deemed to be currently unrefuted, and refuter makes attempt to provide a refutation for that statement, on the basis of a set of counter-assumptions. We can call this a REFUTATION-ATTEMPT, which, informally, is a sequence of arguments that can be utilized by "refuter" to suggest that, given a set of (assumed deniable) premises, the final formula is refutable. The success of a refutation attempt, therefore, depends upon the existence of a refutation for each counter-assumption, and at each step of the argument.

For example, take the following interaction. "Prover" asserts a conjunction, $\alpha \wedge \beta$, so it is incumbent upon prover, to provide a proof attempt giving some sort of reason or evidence in support of both α and β. Refuter, on the other hand, challenges $\alpha \wedge \beta$ by providing a refutation attempt giving some sort of reason or evidence refuting either α or β. Exactly the reverse is the

[29]Further details are in [49].

case if the formula in question is a disjunction. So, we can interpret the relationship between L_I and L_C in terms of tests, where a test of α^+ is just a refutation-attempt of form α^-, and a test of α^- is just a proof-attempt of form α^+:

- Testing $\alpha^+ \wedge \beta^+$ involves testing α^+ or testing β^+.
- Testing $\alpha^+ \vee \beta^+$ involves testing α^+ and testing β^+.
- Testing $\alpha^- \wedge \beta^-$ involves testing α^- and testing β^-.
- Testing $\alpha^- \vee \beta^-$ involves testing α^- or testing β^-.
- Testing $\alpha^+ \Rightarrow \beta^+$ involves testing for a function that maps each test of α^+ into a test of β^+.
- Testing $\beta^- \Leftarrow \alpha^-$ involves testing for a function that maps each test of α^- into a test of β^-.

In order that these tests do not simply collapse together, there must be no restriction upon simultaneous proof and refutation attempts of the same formula α, bearing in mind that there is a syntactic distinction between α^+ in L_I, and α^- in L_C, which we are interpreting as saying that α is being played by "prover", and by "refuter", respectively. This syntactic separation is, therefore, non-arbitrary in the sense that it reflects distinct uses for the two logics as structures for proofs and refutations, respectively, which has the effect that the resultant structure can be understood as a symmetric logic, capable of non-trivially dealing with the simultaneous consideration of proof and refutation attempts. This does not mean that we allow "contradictions" of the form $\alpha^+ \wedge \alpha^-$, since α^+ and α^- exist in separate structures. Rather, we allow for the simultaneous existence of a potential proof of α^+ and refutation of α^- without also allowing for "direct" contradiction. In addition we have two, more standard, notions of negation defined in terms of implication and co-implication, respectively:

- $\neg_I \alpha \leftrightarrow \alpha \Rightarrow 0$
- $\neg_C \alpha \leftrightarrow 1 \Leftarrow \alpha$.

But, in distinction to a typical understanding of negation, and since we already have direct proof (in L_I) and direct refutation (in L_C) available, these will be interpreted as "weak negations" which, attached to a formula α, expresses the idea that an attempted proof (refutation) of α "does not go through".[30] So, $\neg_I \alpha$ corresponds to the assumption of α leading to a non-proof, and so returns an unproved conjecture; and $\neg_C \alpha$ corresponds to the assumption of α leading to a "non-refutation", so similarly returning an unrefuted conjecture. These weaker than typical negation operators allow that α remains an open problem, and so we can make sense of the status of conjectures that are under consideration, whilst also allowing that a proof- or refutation-attempt fails. Weakly-negated formulas obviously fail typical monotonicity conditions, but as [46, §2.4] point out, adding such an operator to standard intuitionistic logic would be conservative since its existence has no impact on the usual interpretation of all other connectives. The same is

[30] This is similar to the empirical negation discussed in [8, 46].

true for co-intuitionistic logic (as structure of refutation), as can easily be checked. As such, these can be expressed in the following BHK-style clauses:

- c is a proof attempt of $\neg\alpha$ if c is a proof attempt of α such that c does not go through .
- c is a refutation attempt of $\neg\alpha$ if c is a refutation attempt of α such that c does not go through .

So, standard characterisations of negation in L_I and L_C mark the that a proof or refutation attempt has not been successful, but this does not bring with it a refutation, or proof, in the dual logical structure.

Whilst this provides something like an informal interpretation of the relationship between the two structures, we may rightly ask how to formalise proofs and refutations that are not merely attempts, but that have "passed" the relevant tests, and have been conclusively established. So, in addition to proof and refutation attempts, it is also the case that certain proofs and refutations may be *conclusive*.[31] It is natural to take a conclusive proof (refutation) to mark the end-point of a questioning process: "is α provable?" (in L_I), "is α refutable?" (in L_C). At this end-point, we have a conclusive result to the process of dialogue, and which puts an end to all possible doubts regarding α. Furthermore, and in distinction with proof and refutation attempts, it seems correct to say that the kind of things that have been conclusively proved, or conclusively refuted, are established TRUE or established FALSE. That is, whilst a conjecture is not the sort of thing that we want to say is true or false, at the end of the process of inquiry, and we have a conclusive proof (refutation) of α, which we may then consider to be a true, or false, proposition. This is simply an extension of the intuitionistic proposal that a proposition is identified with its set of proofs. Here, a proposition is identified with its set of conclusive proofs or refutations. Then, for a proposition, α, we should be able to say of α that it "holds", that α is true (false), and so on. So, we can think of a proposition as a conclusive solution, which terminates attempts to answer that conjecture. I will assume that the distinction between, for example, proof attempts and conclusive proof is not simply a matter of "logic", following Hintikka [19], for example. Nonetheless, it seems correct to say that, once α is refuted conclusively, it can not be proved; once α is proved conclusively, it can not be refuted.

4. Collapse and cartesian closed categories

An immediately apparent issue concerns how we are to construe the exact *formal* relationship between the two structures, aside from noting their duality. It is simplest to see where this is problematic from the point of view of adding an operator equivalent to $\Leftarrow$ to a Cartesian closed category. This is also useful as preliminary structure for understanding the construction of topoi in the following section.

[31] The notion of a "conclusive defense" is discussed in [21].

Definition 4.1. (Cartesian closed category (CCC)) A CCC is a category $\mathbb{C}_{\mathbb{H}}$ with a terminal object, finite products and exponentials such that, for every object A of $\mathbb{C}_{\mathbb{H}}$, the functor $- \times A : \mathbb{C}_{\mathbb{H}} \to \mathbb{C}_{\mathbb{H}}$ has a right adjoint [1, 24].

Let us briefly define initial and terminal objects in $\mathbb{C}_{\mathbb{H}}$.

Definition 4.2. An object 1 is terminal in $\mathbb{C}_{\mathbb{H}}$ if, for any object A of $\mathbb{C}_{\mathbb{H}}$, there is a unique arrow $\Box_A : A \to 1$. An object 0 is initial if for any object A of $\mathbb{C}_{\mathbb{H}}$, there is a unique arrow $\circ_A : 0 \to A$.

We can also define exponentials as follows:

Definition 4.3. (Exponential) An exponential of objects A and B, denoted B^A, is defined by the arrow $\in_{A,B}$: $B^A \times A \to B$ which satisfies the following property: for any object C, and any arrow $f : C \times A \to B$, there is a unique arrow $g : C \to B^A$ as in the following diagram:

$$\begin{tikzcd} C \times A \arrow[r, "f"] \arrow[d, "g \times id_A"'] & B \\ B^A \times A \arrow[ur, "\in_{A,B}"'] & \end{tikzcd}$$

The arrow $\in_{A,B}$ is usually called evaluation.

If we allow that, in $\mathbb{C}_{\mathbb{H}}$ we also have finite co-products and initial object, then we have a Heyting algebra as a category. As such, it is possible to interpret intuitionistic calculus in $\mathbb{C}_{\mathbb{H}}$ much as above. Objects of $\mathbb{C}_{\mathbb{H}}$ interpret formulas α, β; morphisms $f : A \to B$ model proof attempts $\alpha \vdash_I \beta$; composition models cut; products model conjunction; co-products model disjunction; initial and terminal objects are $0, 1$ respectively; exponentiation models implication. On the latter, for example, if we allow that B^A interprets $(\alpha \Rightarrow \beta)$ in an intuitionistic calculus, then it is obvious that the evaluation morphism is equivalent to modus ponens.

We briefly consider co-products.

Definition 4.4. (Co-product) A co-product of A and B is an object $A \oplus B$, with morphisms:

$i_{A,B} : A \to A \oplus B$;

$i'_{A,B} : B \to A \oplus B$;

such that, for any C, and any morphisms $f : A \to C$, and $g : B \to C$, there is a unique arrow $h : A \oplus B \to C$ as in the following diagram:

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "i_{A,B}"'] & A \oplus B \arrow[u, "h"'] & B \arrow[ul, "g"'] \arrow[l, "i'_{A,B}"] \end{tikzcd}$$

The morphisms $i_{A,B}$ and $i'_{A,B}$ are "injections", which ensure the disjunction property in intuitionistic logic.

The general idea is to construe morphisms in $\mathbb{C}_{\mathbb{H}}$ as proofs in L_I, and with 1 understood as "proved" rather than "true" as is commonly the case. This ensures that the structure can be understood syntactically, so that morphisms such as $f : 1 \to A$ interpret theorems of L_I.

Consider now what happens when we attempt to add an operator such as $\Leftarrow$ to C_H. To do so, we introduce co-exponentials into $\mathbb{C}_{\mathbb{H}}$ by means of dualizing the above definition, such that every $- \oplus A : \mathbb{C}_{\mathbb{H}} \to \mathbb{C}_{\mathbb{H}}$ has left adjoint. As is obvious, this is equivalent to the definition of co-implication given for Co-Heyting algebras.

Definition 4.5. (Co-exponential [7]) A co-exponential of objects A and B, denoted B_A, is defined by the arrow $\ni_{A,B}$: $B \to B_A \oplus A$ which satisfies the following property: for any object C, and any arrow $f : B \to C \oplus A$, there is a unique arrow $g : B_A \to C$ as in the following diagram:

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "\ni_{A,B}"'] & C \oplus A \\
& B_A \oplus A \arrow[u, "h \oplus id_A"']
\end{tikzcd}$$

Let us note that, as discussed above, the co-exponential (insofar as it is understood to interpret co-implication) does not suffice as an implication operator. Nonetheless, it is instructive to consider the morphism $\ni_{A,B}$, which we call co-evaluation, if we allow B_A to interpret $(\beta \Leftarrow \alpha)$. Co-evaluation gives us $\beta \leq (\beta \Leftarrow \alpha) \vee \alpha$.

Of course, this also allows us to construct two negations by means of exponentials and co-exponentials within C_H. In this vein, attempts to allow L_I and L_C to "cohabit" in the same structure have been explored by means of bi-intuitionistic calculi, which include both intuitionistic and co-intuitionistic negation. Algebraically, these are expressed by bi-Heyting structures [37] combining Heyting and co-Heyting algebras, which are equivalent to C_H with co-exponential. Importantly, Crolard [7] shows that, for any CCC, if we add co-exponential, it will collapse to a single partial order. This is related to Joyal's lemma, which proves any CCC that is self-dual is a pre-order. Importantly for later, this can be strengthened since, whenever the self-duality is due to a dualizing object (such as a negation operator), then the posetal reflection of the preorder will be a Boolean algebra. This is due to the fact that, whenever a CCC contains exponentials, initial object is strict (so $0 \times 0 \cong 0$). Then, by presence of co-exponentials, we know that $A \oplus - : \mathbb{C}_{\mathbb{H}} \to \mathbb{C}_{\mathbb{H}}$ has left adjoint, which distributes over product as follows: $A \cong A \oplus 0 \cong A \oplus (0 \times 0) \cong (A \oplus 0) \times (A \oplus 0) \cong A \times A$, and this entails that, for any two maps into A (providing they have the same domain) are equivalent.[32]

[32] This follows an informal proof given in correspondence by Peter Johnstone available at: http://permalink.gmane.org/gmane.science.mathematics.categories/7045, and [7] provides a formal proof with discussion.

5. Dual topoi and coherence

To pinpoint the exact mechanisms of collapse both algebraically and logically, we now construct separate topos-theoretic structures for L_I and L_C. In doing so, it will become possible to control collapse to the extent that we can utilize it in our construction.

A topos can be understood as an extension of a CCC with a subobject classifier, which means that it is particularly useful for providing a structural analysis of propositional logics. Following the motivation given in [1], we may think of a topos as a kind of category consisting of '(i) objects with some arbitrary structure, i.e., with morphisms between them, and (ii) "everything that can be constructed from these by logical means"' [1, p.223]. In [32] (and developed in [13]), it is shown that it is possible to construct topoi whose internal logical structure is paraconsistent. This is in sharp distinction with the commonly-held [15, 25] assumption that the internal logic of any elementary topos is intuitionistic.[33] Further still, it is argued in [13] that the internal logic of a topos is not fully determined by its mathematical structure, but rather, considered as an abstract structure, a topos can support a variety of internal logics. We make use of this here, by constructing dual topoi for proofs and refutations, before identifying the manner in which they collapse to a Boolean topos so that it may be controlled.

Definition 5.1. (Subobject classifier [25]) In a category C with finite limits, a subobject classifier is a monic: $t : 1 \to \Omega$, where Ω is an object of C, such that, for every object A and any monic $m : S \to A$ in C, there is a unique arrow $\theta : A \to \Omega$ which forms the pullback square:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "t"] \\ A \arrow[r, "\theta"'] & \Omega \end{tikzcd}$$

Then, we call Ω the subobject classifier of C, and θ is the classifying map of m.

We are now in a position to define a topos.

Definition 5.2. (Topos) A topos T is a category with finite limits and colimits; exponentials; a subobject classifier. This is equivalent to the simpler definition that T is a topos whenever T is a category with terminal object, pullbacks, exponentials, and a subobject classifier. In a topos, a proposition is simply a morphism $1 \to \Omega$.

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[33] [13] also points out that this has also been taken to provide support for the far stronger suggestion that the 'universal, invariant laws of mathematics are those of intuitionistic logic', in [2].

Theorem 5.3. [25, p.198ff] *For any topos T, and for any object A in T, the subobjects of A, Ω^A, form a Heyting algebra.*[34]

Proof. We first provide categorial definitions of the standard connectives of a Heyting algebra in the context of T by means of morphisms of the subobject classifier.

Conjunction, is defined as the characteristic morphism of $\langle t, t\rangle : 1 \to \Omega \times \Omega$, so that $\Omega \times \Omega \overset{\wedge}{\to} \Omega$, makes the following a pullback diagram:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\langle t,t\rangle"'] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

Disjunction, is defined as the characteristic morphism of $[\langle t, id\rangle, \langle id, t\rangle]$, so that $\Omega \times \Omega \overset{\vee}{\to} \Omega$, makes the following a pullback diagram:

$$\begin{tikzcd} \Omega \oplus \Omega \arrow[r] \arrow[d, "{[\langle t,id\rangle,\langle id,t\rangle]}"'] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "\vee"'] & \Omega \end{tikzcd}$$

Implication, is defined as the characteristic morphism of $e : \leq \overset{e}{\to} \Omega \times \Omega$ (where $\leq$ is the partial order formed by the elements of Ω), which is the equalizer of $\wedge$ and the canonical projection, π_1, on the first component of the product, i.e. $\leq \overset{e}{\to} \Omega \times \Omega \underset{\pi_1}{\wedge} \Omega$, so that $\Omega \times \Omega \overset{\Rightarrow}{\to} \Omega$, makes the following a pullback diagram:

$$\begin{tikzcd} \leq \arrow[r] \arrow[d, "e"'] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "\Rightarrow"'] & \Omega \end{tikzcd}$$

Note that this mirrors the fact that in a Heyting algebra, $\alpha \leq \beta \leftrightarrow \alpha \wedge \beta = \alpha$.

Let us define the morphism $f_T : 1 \to \Omega$ by means of the initial object 0 by the following diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\ 1 \arrow[r, "f_T"'] & \Omega \end{tikzcd}$$

[34] As is typical, we may slightly abuse notation say that a subobject may be a member of Ω, since this makes little difference in practice (in the context of propositional logic), though to be more precise we should say that the equivalence class for a subobject is a member of Ω.

Then, negation is defined as the characteristic morphism of f_T, so that $\Omega \xrightarrow{\neg} \Omega$ makes the following a pullback diagram:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "f_T"'] & 1 \arrow[d, "t"] \\ \Omega \arrow[r, "\neg"'] & \Omega \end{tikzcd}$$

Then, since $\Omega \times \Omega \Rightarrow \Omega$ constructs an exponential object Ω^{Ω}, so we can define a Heyting algebra by the poset structure of Ω. $\square$

Since propositional intuitionistic logic is equivalent to any Heyting algebra, so it is unsurprising that the same is true in a topos T. We can extend the translation into Heyting algebra given above, and interpret a formula α of a formal language, whose set of formulas is S, by means of a topos T, to be denoted as $T \models_T \alpha$, which says that α is equivalent to the morphism t in T.[35] Similarly, $\alpha \models_T \beta$ says that whenever α is equivalent to the morphism t in T, so too is β.[36]

Theorem 5.4. *For any topos T and proposition α, $T \models \alpha$ whenever $\vdash_I \alpha$.*

Proof. By the fact that Ω is a Heyting algebra, and given in [25]. $\square$

Of course, every Boolean algebra is also a Heyting algebra when each element, α of a Heyting algebra is complemented so that $\neg\neg\alpha = \alpha$. In the topos-theoretic setting, we will say that a topos T_B is Boolean whenever Ω forms a Boolean algebra [25, p.270]. This is the case whenever f is complement to t, so that the negation operator $\neg : \Omega \to \Omega$ satisfies $\neg\neg = id$, and, by Diaconescu's theorem, this is also equivalent to a topos having axiom of choice. As example, the topos SET has a subobject classifier which forms a Boolean algebra, where the objects of SET are sets, and morphisms are functions.

Now, consider a topos adequate to a co-Heyting algebra. By analogy with the relationship between T and intuitionistic logic, we require a topos with Ω forming a co-Heyting algebra, and by which it is possible to construct an interpretation of co-intuitionistic logic as logic of refutation. The key distinction between the approach suggested here, and the construction in [12, 13, 32] is that, in the latter, a paraconsistent topos is understood as a complete mathematical universe, which is, at the least, capable of expressing a "full" propositional logic. Here, we take an ordinary topos and a paraconsistent topos to express two "halves" of a logic, with the former dealing with proof attempts, and the latter, refutation attempts.[37] This allows us to deal

[35] Bear in mind that under this interpretation, t does not represent "truth", but "potential proof".
[36] This follows the account given in [1].
[37] It may seem odd to allow intuitionistic logic to range over proofs only, but, as [40] points out, it is incapable of dealing with refutation in any case.

with complications that can they can not, such as the fact that a paraconsistent topos (and logic in general) is incapable of expressing a detachable implication operator for proofs.

As pointed out in [13], there is little reason to interpret the morphism t as indicating "truth", or "proof" as ordinarily is the case, and such an interpretation is a matter of external decision, which is to say that it is not forced upon us by the internal language of a standard topos. So, whilst the construction of a paraconsistent topos may be seen by some as a simple matter of relabelling, this will have significant impact upon the structure of the propositional logic that we interpret in it. For example, the process of constructing a complement topos begins with the subobject classifier, which is renamed "complement classifier", which, as such, is a is a different generalized element than $t : 1 \to \Omega$ in T.

Definition 5.5. (Complement classifier [12, 13, 32]) In a category T_C with terminal object 1, a complement classifier is a monic: $f : 1 \to \Omega$, where Ω is an object of T_C, such that, for every object A and any monic $m : S \to A$ in T_C, there is a unique arrow $\sigma : A \to \Omega$ which forms the pullback square:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "f"] \\ A \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

Then, we call Ω the complement classifier of C, and σ is the classifying map of m.

Definition 5.6. (Complement topos [12, 13, 32]) A complement topos T_C is a category with finite limits and colimits; exponentials; a complement classifier. This is equivalent to the simpler definition that T_C is a complement topos whenever T_C is a category with terminal object, pullbacks, exponentials, and a complement classifier. In a complement topos, a (refutation) proposition is a morphism $1 \to \Omega$.

Theorem 5.7. [12, 13, 32] *For any complement topos T_C, and for any object A in T_C, the subobjects of A, Ω^A, form a co-Heyting algebra.*

Proof. We first provide categorial definitions of the standard connectives of a co-Heyting algebra in the context of T_C by means of morphisms of the subobject classifier. Conjunction, is defined as the characteristic morphism of $[\langle f, id\rangle, \langle id, f\rangle]$, so that $\Omega \times \Omega \xrightarrow{\wedge} \Omega$, makes the following a pullback diagram:

$$\begin{tikzcd} \Omega \oplus \Omega \arrow[r] \arrow[d, "{[\langle f,id\rangle,\langle id,f\rangle]}"'] & 1 \arrow[d, "f"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

Disjunction, is defined as the characteristic morphism of $\langle f, f\rangle : 1 \to \Omega \times \Omega$, so that $\Omega \times \Omega \xrightarrow{\vee} \Omega$, makes the following a pullback diagram:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\langle f,f \rangle"'] & 1 \arrow[d, "f"] \\ \Omega \times \Omega \arrow[r, "\vee"'] & \Omega \end{tikzcd}$$

Co-implication is defined as the characteristic morphism of $\overline{e} : \geq \xrightarrow{\overline{e}} \Omega \times \Omega$, which is the equalizer of $\vee$ and the canonical projection, π_1, on the first component, i.e. $\geq \xrightarrow{\overline{e}} \Omega \times \Omega \underset{\pi_1}{\overset{\vee}{\rightrightarrows}} \Omega$, so that $\Omega \times \Omega \xrightarrow{\Leftarrow} \Omega$, makes the following a pullback diagram:

$$\begin{tikzcd} \geq \arrow[r] \arrow[d, "\overline{e}"'] & 1 \arrow[d, "f"] \\ \Omega \times \Omega \arrow[r, "\Leftarrow"'] & \Omega \end{tikzcd}$$

Note that this mirrors the fact that in a co-Heyting algebra, $\alpha \geq \beta \leftrightarrow \beta \vee \alpha = \beta$.

Let us define the morphism $t_{T_C} : 1 \to \Omega$ by means of the initial object 0 by the following diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "f"] \\ 1 \arrow[r, "t_{T_C}"'] & \Omega \end{tikzcd}$$

Then, negation is defined as the characteristic morphism of t_{T_C}, so that $\Omega \xrightarrow{\neg} \Omega$ makes the following a pullback diagram:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "t_{T_C}"'] & 1 \arrow[d, "f"] \\ \Omega \arrow[r, "\neg"'] & \Omega \end{tikzcd}$$

Then, since $\Omega \times \Omega \Rightarrow \Omega$ constructs an exponential object Ω^{Ω}, we can define a co-Heyting algebra by the poset structure of Ω. □

Since propositional co-intuitionistic logic is equivalent to a co-Heyting algebra, it is unsurprising that the same is true in T_C. We can extend the translation into co-Heyting algebra given above, and interpret a formula α of a formal language, whose set of formulas is S, by means of T_C, to be denoted as $T_C \models_{T_C} \alpha$, which says that α is equivalent to the morphism f in T_C.[38] Similarly, $\alpha \models_{T_C} \beta$ says that whenever α is equivalent to the morphism f in T_C, so too is β.[39]

[38] As above, bear in mind that f does not represent "falsity", but "proof attempt".
[39] This follows the account given in [1].

Theorem 5.8. *For any complement topos T_C and proposition α, $T_C \models_{T_C} \alpha$ whenever $\dashv_C \alpha$.*

Proof. By the fact that Ω is a co-Heyting algebra, and by duality with the proof for Heyting algebra and ordinary topos [12]. □

As is obvious, a complement topos is a natural structure for articulating co-intuitionistic logic as a logic of refutation, just as a topos is a natural structure for articulating intuitionistic as a logic of proof. For example, the following two properties hold in the respective topoi.

Corollary 5.9. *The disjunction property holds in T* [15]*; the conjunction property holds in T_C.*

Proof. In Intuitionistic logic, we have the disjunction property: $\vdash_I \alpha \vee \beta$ iff $\vdash_I \alpha$ or $\vdash_I \beta$; in co-Intuitionistic logic (as refutation) we have the conjunction property: $\dashv_C \alpha \wedge \beta$ iff $\dashv_C \alpha$ or $\dashv_C \beta$. In the topoi as defined above, these hold by definition of disjunction in T and conjunction in T_C, respectively. Whenever $\alpha \vee \beta$ is equivalent to the morphism t in T, then so is the morphism α or the morphism β. Whenever $\alpha \wedge \beta$ is equivalent to the morphism f in T_C, then so is the morphism α or the morphism β. □

As is typically the case, the intuitionistic disjunction property is connected to its constructive nature, since law of excluded middle is not a theorem. This is mirrored in co-intuitionistic logic, as constructive logic of refutation, where law of non-contradiction is not a theorem. Moreover, just as the simultaneous proof of α and $\neg\alpha$ in L_I would trivialize the logic such that it is possible to prove any formula, so the simultaneous refutation of α and $\neg\alpha$ in L_C would trivialize the logic such that it is possible to refute any formula. In fact, these follow from the typical conditions on initial objects, which, if interpreted as "non-proved" in T, says that from "non-proved" everything follows; if interpreted as "non-refuted" in T_C, says that from "non-refuted" everything follows.

Corollary 5.10. *Intuitionistically, implication is an operator for which deduction theorem holds, such that $\alpha \vdash_I \beta \leftrightarrow \vdash_I \alpha \Rightarrow \beta$. In co-intuitionistic logic, co-implication is an operator for which dual-deduction theorem holds, such that $\alpha \dashv_C \beta \leftrightarrow \dashv_C \beta \Leftarrow \alpha$. Both of these are the case in T and T_C (respectively), by definition.*

As shown by Goodman [16], it is not possible to define an operator in L_C by means of $\wedge, \vee, \neg, \Leftarrow$ that obeys deduction theorem. Furthermore, whilst $\Leftarrow$ will not satisfy modus ponens, it is simple to derive the dual, refutation, form of *modus ponens*:

$$\dfrac{\overset{[\text{ass.}]}{\dashv_C \alpha \Leftarrow \beta} \qquad \dfrac{\overset{[\text{ass.}]}{\dashv_C \beta} \qquad \overset{[\text{Identity}]}{\alpha \dashv_C \alpha}}{\alpha \Leftarrow \beta \dashv_C \alpha}\,(\Leftarrow L)}{\dashv_C \alpha}\,(\text{Cut})$$

If we were interested in L_C (and T_C) as a construction ranging over proofs, then the lack of a definable operator for which deduction theorem holds would be problematic, since it would suggest that such a logic is incapable of defining a decent implication.[40] The reason that it is impossible to define the respective operators inside the opposite structure is obviously connected with the collapse to a preorder when co-exponential is added to CCC. But, in this context, we can go further still, and show that, any attempt to construct a topos whose internal logic supports both intuitionistic and co-intuitionistic logic will collapse to a topos with Boolean algebra such as SET. Consider this in terms of the logical structure first.

Theorem 5.11. *Any attempt to extend L_I with an operator for which dual-deduction theorem holds, or L_C with an operator for which deduction theorem holds is impossible.*[41]

Proof. We consider L_I only (the proof for L_C is analogous). Say that we attempt to add an operator, $\Leftarrow$, for which dual deduction theorem holds in L_I, so that $\alpha \vdash_I \beta \leftrightarrow \beta \Leftarrow \alpha \vdash_I$. We take as assumption (since it is well-known), that the counter-theorems of L_I are equivalent to the counter-theorems of classical logic, and that $\Leftarrow$ is definable classically as $(\alpha \Leftarrow \beta) := (\alpha \wedge \neg\beta)$. Since we have $\alpha \vdash \alpha$ both classically and in L_I, we also have $\alpha \Leftarrow \alpha \vdash$ in both. Classically, we can rewrite any α with its double negation, so, there, we have $\neg\neg\alpha \Leftarrow \alpha \vdash$ (and in L_I by the assumption above). Therefore, we can derive $\neg\neg\alpha \vdash \alpha$ in both, which is not possible in L_I. The reverse is easily proved for $L_C + \Rightarrow$.[42] $\square$

The problems arise whenever we attempt to put together the calculi whilst also requiring that deduction and dual-deduction theorem hold. For example, it is possible to formulate deduction and dual-deduction theorems in rule-form and add them to either L_I and L_C, so that we have the following: for any α, β, if there exists a derivation of $\vdash_I \beta$, which is either an axiom or

[40]The point is arguable. For example, [12] discusses the suggestion in [32] that, in fact, this is not a problematic deficiency since mathematics depends less on an object-language implication operator, than it does a deducibility relation, which can be constructed in terms of the ordering on the complement classifier. I will not take a stance on this here, since the current construction does not require such an operator.

[41]A similar proof is given in [52]

[42]It is possible to simply add to L_C a kind of implication (or co-implication to L_I) connective [52]. But, neither modus ponens nor deduction theorem will unrestrictedly hold for it, and we lose constructivity in the form of disjunction property for positive entailment and conjunction property for negative entailment (there is also no way to do so conservatively for first-order logic).

a sequent of the form $\vdash_I \alpha$, then $\alpha \vdash_I \beta$; if there exists a derivation of $\dashv_C \beta$, which is either an axiom or a sequent of the form $\dashv_C \alpha$, then $\alpha \dashv_C \beta$. But, it is difficult, then, to prevent classical logic following closely behind, since in standard settings this makes it possible to derive both double negation introduction and elimination (as discussed in [7]).[43]

Nonetheless, whilst it is certainly *possible*, if tricky, for intuitionistic and co-intuitionistic logic to be supported within the same structure, both logically, and algebraically, this is not the case for topos-theoretic interpretations.[37] provides an overview of bi-Heyting algebras. For example, it is not possible for a standard topos T with Ω forming a Heyting algebra to also have Ω form a co-Heyting algebra unless the algebra formed is Boolean.

Lemma 5.12. *In a topos T, the characteristic morphism of $0 \to 1$ is $f_T : 1 \to \Omega$; in a complement topos T_C, the characteristic morphism of $0 \to 1$ is $t_{T_C} : 1 \to \Omega$. Whenever a topos T also has a complement classifier such that both $t : 1 \to \Omega$ and $f : 1 \to \Omega$ classify, then T is Boolean. Whenever a complement topos T_C also has a classifier such that both $f : 1 \to \Omega$ and $t : 1 \to \Omega$ classify, then T_C is Boolean.*

Above, we defined a Boolean topos, T_B whenever Ω forms a Boolean algebra, which is the case whenever $\neg\neg = id$; equally, this is the case whenever $\neg f = t$ and $\neg t = f$, so, for any α $\neg\neg\alpha = \alpha$. Equivalently, in any topos T, $1 \oplus 1$ is definable by presence of co-product, and, by definition $t, f : 1 \to \Omega$, so we have $1 \oplus 1 \to \Omega$. Whenever the latter morphism is an isomorphism, the topos T is Boolean since every proposition α in T is equivalent with either t or f. The above lemma is equivalent to saying that the topos forms subobjects with Heyting algebra for provability, and co-Heyting for refutability. Letting $f_t = f$ (in T_C), and $t_{T_C} = t$ (in T), we know by respective definitions of negation that both $\alpha \leq \neg\neg\alpha$ and $\neg\neg\alpha \leq \alpha$, by which it follows that $\alpha = \neg\neg\alpha$ for every α. In other words, whenever $\neg\alpha$ is equivalent to the morphism t, then α is equivalent to the morphism f, and vice-versa, so we have $\neg\neg = id$.[44]

[43] [17] constructs a combined logic using a display calculus, which is capable of dealing with both proofs and refutations on a par. In order for cut-elimination, however, it must be possible to trace the direction (positive or negative) of entailment, and so there is a metalogical decision that, for any sequent, its direction must go in one of the two ways. Consider also the metalogical relationship between L_I and L_C. By duality, we know that (co-)theorematically, whenever it is the case that $\vdash_I \alpha$, it is also the case that $\dashv_C \alpha$, so that L_C is something like the metalogical negation of the system L_I. This leads us to suppose that, for any α, either α is proved in L_I, or its dual is refuted in L_C, which is just a metalogical form law of non-contradiction. It is also worth pointing out that any symmetric calculus also requires restrictions that reflect the restriction to single succedents for positive entailment, and single antecedents for negative entailment, since if the deduction theorem is not so restricted, it is simple to derive excluded middle as theorem (from $\alpha \Rightarrow 0 \vdash_I \neg\alpha$, derive $\vdash_I \alpha, \neg\alpha$, and then use the right disjunction rule). This is tricky to achieve (particularly in a non-arbitrary way), since it wreaks havoc with structural rules.

[44] For example, [12] points out that in effect, SET, therefore always has both classifiers.

Since the internal logic of a topos is defined by the algebra that is formed by Ω, the internal logic of a Boolean topos is classical [15, ch.7]. This gives us the following corollary:

Corollary 5.13. *In a topos-theoretic setting, L_I and L_C can not co-habit, since any attempt to combine L_I and L_C (or T and T_C) will collapse to classical logic (or Boolean topos).*

6. Harnessing collapse for dialogue between proofs and refutations

In order to prevent the automatic collapse of T and T_C to a Boolean topos, we have syntactically separated the two structures, distinguishing the objects from which they are built.[45] This has the consequence that certain "formulas" are barred. For example, $\neg_I\alpha^-$ and $\neg_C\alpha^+$ are not wff's, and so neither is $\neg_I\neg_C\alpha^{+,-}$. So, it is possible that proof attempts exist neither for α nor $\neg_I\alpha$; and that refutation attempts exist neither for $\alpha^\perp$ nor $\neg_C\alpha^\perp$. Nor is there any restriction on the simultaneous consideration of proof and refutation attempts as discussed above. As such, there is nothing in the above topoi T and T_C, that requires the following to hold in general:

- $T \models_T \alpha^+$ or $T_C \models_{T_C} \alpha^-$
- $T \not\models_T \alpha^+$ or $T_C \not\models_{T_C} \alpha^-$

From the above discussion, we also say that once a formula α^- is refuted *conclusively*, then α^+ can not be proved; once α^+ is proved conclusively, α^- can not be refuted. This gives us a distinction between proof and refutation *attempts*, and *conclusively* proved (refuted) formulas that we can make use of here.

In a typical constructive logic such as L_I, we say that a formula α is decidable iff $\alpha \vee \neg\alpha$ holds for α; we say that a formula α is stable iff $\neg\neg\alpha$ implies α. Neither of these is a theorem of L_I, though they hold in certain contexts. By analogy with this, in the current structure we shall say that a formula α is *determined* whenever there exists a *conclusive* proof of α^+, or a *conclusive* refutation of α^-. Moreover, as in the interpretation given in §3, it seems appropriate to say that some α for which we have a conclusive proof (refutation), that it holds true (false).[46] What is important about the set of formulas for which a *conclusive proof or refutation* exists is that we are then in a position to *know*: the proof of α^+ *rules out* any further refutation attempts of α^-; or the refutation of α^- *rules out* any further proof attempts of α^+. As such, over these (and only these formulas), it becomes possible to define an *external* relationship between formulas of S and S^d by means of a kind of metalinguistic negation operation, since, if, for some $\alpha^{+,-}$, α^+ is conclusively

[45] Whilst differing in both interpretation and formalization, a similar technique is suggested in [4, 31].

[46] Take, for example, a domain concerning rationality such as scientific reasoning or argumentation. There, it is not uncommon to have reason (or some evidence) to consider potential proofs of α, whilst also attempting to "test" α as defined above.

proved, then α^- is *not* conclusively refuted; if α^- is conclusively refuted, then α^+ is *not* conclusively proved. As such, we can define the following *external* negation between L_I and L_C:

Definition 6.1. For all and only those formulas of S and S^d which are conclusively proved (in L_I) or conclusively refuted (in L_C), the following metalinguistic negation operation, $\sim$, to hold between formulas of the logical structures:

- $\sim \alpha^+ := \alpha^-$
- $\sim \alpha^- := \alpha^+$
- $\sim\sim \alpha^{+,-} = \alpha^{-,+}$

Since $\sim$ is an external negation, it negates a formula as a whole, rather than some component of that formula. So, here, it carries the simple meaning that, for example, $\sim (\alpha^+ \wedge \beta^+)$ is equivalent to $(\alpha^- \wedge \beta^-)$. In English, this just says that, whenever there exists a conclusive refutation of "α and β", we also know that there can *not* exist a conclusive proof of "α and β".

We can then use $\sim$ to simulate a metalinguistic (i.e. external) relationship between the topos-theoretic structures introduced above. Let us call this relationship coherence:

Definition 6.2. Say that T and T_C are coherent iff it is the case that whenever $T \models_T \alpha$ then $T_C \not\models_{T_C} \alpha$, and whenever $T_C \models_{T_C} \alpha$ then $T \not\models_T \alpha$, and, for every α, either $T \models_T \alpha$ or $T_C \models_{T_C} \alpha$, and either $T \not\models_T \alpha$ or $T_C \not\models_{T_C} \alpha$. So, $\sim \alpha^+$ is equivalent to saying that whenever $T_C \models_{T_C} \alpha$ then $T \not\models_T \alpha$; $\sim \alpha^-$ is equivalent to saying that whenever $T \models_T \alpha$ then $T_C \not\models_{T_C} \alpha$, so the above clauses defining $\sim$ hold for coherent elements of T and T_C.

It is simple to see from the above, and by the fact that $\sim\sim = id$, that it is possible to construct a topos (T_B) containing only *determined* formulas, with $\sim f = t$ and $\sim t = f$, so, for any α, $\sim\sim \alpha = \alpha$. In other words, the following holds:

Corollary 6.3. *Whenever T and T_C are coherent, they can be represented by a Boolean topos T_B.*

This also means that the internal logic of a Boolean topos (i.e. classical logic), can be simulated by considering T alongside T_C, but *only* when they are in a relationship of coherence. In addition, we also have the result that, for all and only those formulas which are conclusively proved (in L_I) or conclusively refuted (in L_C), they may be interpreted in a Boolean topos such that $T_B \models \alpha$ or $T_B \models\sim \alpha$ for each α.

This is analogous to the well-known result that a double-negation morphism, $\sim\sim : \Omega \to \Omega$, on subobjects of a topos T, defines a closure operation called a Lawvere-Tierney topology (or double negation topology) on T. Then, the associated topos of sheaves, $T_{\sim\sim} \hookrightarrow T$ that corresponds to the double negation topology is a Boolean topos (i.e. $T_{\sim\sim} := T_B$).[47] As such, the topos

[47]Proofs and further discussion may be found in detail in [26], and are also given by Todd Trimble here: https://ncatlab.org/nlab/show/Heyting+algebraToBooleanAlgebras.

of sheaves $T_{\sim\sim}$ is a model of classical logic. Here, we are simulating this construction by taking only those formulas that exist in the dialogue topoi, T and T_C that are conclusively proved (refuted). By the above definition of coherence over these formulas, we can then define a double-negation morphism, $j =\sim\sim= \Omega \to \Omega = id$, on the respective subobjects of T and T_C such that the determined subobject $J \to \Omega$ classified by j includes only the set of conclusively proved and refuted formulas. The subtopos $T_{\sim\sim}$ of T and T_C formed in this way is obviously Boolean by the fact that $\sim\sim= id$.

The machinery of this interpretation may be clarified by means of a hybrid Kripke-Joyal semantics for presheaf topos, complement topos, and Boolean topos. For sake of simplicity, let us use the standard Kripke-style construction of a model, with the distinction being that we have a hybrid structure combining objects of a Boolean topos with objects of "dialogue" topoi above them.[48] Only objects of a Boolean topos will be counted as satisfying formulas in the sense that formulas there are given a definite truth-value, which is retained by persistence (at each stage of the construction of the relevant subobject). Objects of dialogue topoi, on the other hand, are self-contained and do not "reach-into" other stages at all, so they represent a space of dialogue existing prior to the forcing of a definitive truth-value. So, we shall consider a proposition as a function from stages of reasoning as represented by an object A_i of a Boolean topos T_B.

Definition 6.4. Take the two languages S and S^d, with the three topoi, T, T_C, T_B as defined in the previous section.[49] Then, since we are primarily interested in constructing an interpretation $\mathcal{M} : S, S^d \to T_B$, we take an object A_i of T_B, α a formula of S, S^d and $A_i \xrightarrow{a_i} X_i, 1 \leq i \leq n$, then we call the morphism a_i the generalised element of X_i at stage A_i. We also have, in addition, interpretations $S \to T$, and $S^d \to T_C$, which we will define over objects A_i of T and T_C.
We can then define the forcing relations $A_i \models^+ \alpha$ and $A_i \models^- \alpha$, and say that α positively or negatively holds at stage A_i, respectively, for T_B. The relation $A_i \models^+ \alpha$ also holds for T, whilst the relation $A_i \models^- \alpha$ holds for T_C, and these are taken to denote proof and refutation *attempts*, respectively.
We can also define a poset, $\leq$, as per usual, over stages $\mathcal{S}$ such that $\leq$ is transitive and reflexive over $\mathcal{S}$. The idea is that we have "Boolean" stages,

[48] For an alternative presentation, and for further details of the semantics for an ordinary topos, T, and Boolean topos, T_B, see [26, p.302ff] and also [28, p.783ff], the latter of which the following definition draws upon.

[49] We should say something here about the syntax of the Boolean topos T_B, which we are loosely specifying in the following as simply S, S^d. This looseness is not harmful since it is possible to define all of the syntax of both within a topos whose internal language is Boolean, and this follows from the collapse result in the previous section. For example, $\neg_I$ is definable by means of $\Rightarrow$ and 0 as usual, and $\neg_C$ by $\Leftarrow$ and 1, as they were for L_I and L_C, respectively. The difference here is that they collapse into each other by the fact that $\neg_I \neg_C = id$ in T_B. As such, and due to the (non-formal) interpretation of conclusive proofs and refutations, it is safe to replace $\neg_I$ and $\neg_C$ in T_B by the single external negation $\sim$ defined above.

$\mathcal{B} \subseteq \mathcal{S}$ which are just those for which a conclusive proof or refutation has been established. At these (and only these) stages, $\models^+$ and $\models^-$ are monotonic for each $A_i \in \mathcal{B}$. In other words, they satisfy the following properties:
If $A_1 \leq A_2$ then $\forall\alpha \in (\mathcal{B}, A_1), \alpha \in (\mathcal{B}, A_2)$; and If $A_1 \leq A_2$ then $\forall\alpha \in (\mathcal{B}, A_1), \alpha \in (\mathcal{B}, A_2)$.
This ensures that, whenever a formula is conclusive proved or refuted it can be interpreted at a Boolean stage, where that "state" remains at every stage upstream, so, if $A_1 \models^+ \alpha$, then $A_2 \models^+ \alpha$; if $A_1 \models^- \alpha$, then $A_2 \models^- \alpha$. This is not, however, the case for stages outside of $\mathcal{B}$, i.e. at stages of dialogue which is not yet terminated such that a formula is conclusively proved, or conclusively refuted.

For any generalised element $A_i \xrightarrow{a_i} X_i$ of an object X_i in a topos, there are rules specifying when this generalised element belongs to a subobject of X defined in terms of the forcing relation at stage A_i. This allows us to consider proof and refutation attempts at stage A_i in T, T_C, whilst also interpreting conclusive proofs and refutations in T_B. The clauses defining compound formulas are as follows, with Boolean stages reaching above themselves by the monotonicity property. Also, note that Boolean stages do not include the internal negations of L_I and L_C, since these are only ever involved in proof and refutation *attempts*, but they do include the external negation, $\sim$, as defined above, whilst the stages that are non-Boolean do not.

Definition 6.5. (Compound formulas (dropping superscripts throughout)): $S \to T$:

- $[\wedge] A_i \models^+ (\alpha \wedge \beta)$ iff $A_i \models^+ \alpha$ and $A_i \models^+ \beta$
- $[\vee] A_i \models^+ (\alpha \vee \beta)$ iff $A_i \models^+ \alpha$ or $A_i \models^+ \beta$
- $[\Rightarrow] A_i \models^+ (\alpha \Rightarrow \beta)$ iff $A_i \models^+ \alpha$ then $A_i \models^+ \beta$
- $[\neg_I] A_i \models^+ (\neg_I \alpha)$ iff $A_i \not\models^+ \alpha$

$S^d \to T_C$:

- $[\wedge] A_i \models^- (\alpha \wedge \beta)$ iff $A_i \models^- \alpha$ or $A_i \models^-$
- $[\vee] A_i \models^- (\alpha \vee \beta)$ iff $A_i \models^- \alpha$ and $A_i \models^- \beta$
- $[\Leftarrow] A_i \models^- (\beta \Leftarrow \alpha)$ iff $A_i \models^- \alpha$ then $A_i \models^- \beta$
- $[\neg_C] A_i \models^- (\neg_C \alpha)$ iff $A_i \not\models^- \alpha$

$S, S^d \to T_B$

- $[\wedge] A_i \models^+ (\alpha \wedge \beta)$ iff $A_i \models^+ \alpha$ and $A_i \models^+ \beta$
- $[\wedge] A_i \models^- (\alpha \wedge \beta)$ iff $A_i \models^- \alpha$ or $A_i \models^-$
- $[\vee] A_i \models^+ (\alpha \vee \beta)$ iff $A_i \models^+ \alpha$ or $A_i \models^+ \beta$
- $[\vee] A_i \models^- (\alpha \vee \beta)$ iff $A_i \models^- \alpha$ and $A_i \models^- \beta$
- $[\Rightarrow] A_i \models^+ (\alpha \Rightarrow \beta)$ iff $\forall A'_i$ and $A_i \leq A'_i$, if $A'_i \models^+ \alpha$ then $A'_i \models^+ \beta$
- $[\Leftarrow] A_i \models^- (\beta \Leftarrow \alpha)$ iff $\forall A'_i$ and $A_i \leq A'_i$, if $A'_i \models^- \alpha$ then $A'_i \models^- \beta$
- $[\sim]$For all $A_i \in \mathcal{B}, A_i \models^+ \alpha$ iff $A_i \models^- \sim \alpha$
- $[\sim]$For all $A_i \in \mathcal{B}, A_i \models^- \alpha$ iff $A_i \models^+ \sim \alpha$

Definition 6.6. (Satisfaction) We say that a model $\mathcal{M}$ satisfies α positively or negatively, ($\mathcal{M} \models^+ \alpha$ or $\mathcal{M} \models^- \alpha$ respectively) iff $B_i \models^+ \alpha$ for every $B \in \mathcal{B}$

or $B_i \models^- \alpha$ for every $B \in \mathcal{B}$, respectively. Whenever a formula is satisfied positively or negatively, it is called true or false, respectively. A formula is called conclusively proved or refuted iff it is satisfied in all models.

In other words, satisfaction at a single *Boolean* stage implies satisfaction at all stages occurring before it, and so conclusive proof and refutation is equivalent to being forced at all Boolean stages. Moreover, whenever a topos T is Boolean, it can be easily be shown that the semantics defined by the above is just the standard classical set-theoretic semantics [26]. The distinction between the above construction and the latter is that this is a *constructive* form of classical semantics, since it is a requirement, for example, that for $\alpha \vee \sim \alpha$ to be satisfied by T_B, that we have produced a conclusive proof of α, or a conclusive refutation of α. Once a proposition is established true or false is, therefore, dependent upon the state of reasoning at a given time, but once it is so established, by the property of monotonicity, it remains true or false at all subsequent stages.

In other words, by syntactically separating T and T_C we are capable of controlling collapse, allowing for collapse to Boolean topos only in case there exists a conclusive proof or refutation for some formula. This reflects the fact that, in our co-constructive logic, decidability comes at the end of a process of dialogue between attempted proof or refutation of that formula. In this way, we have provided a proof-theoretic semantics for symmetric dialogue between topoi for proofs and refutations. Moreover, far from collapse being problematic for this structure, we utilize it in order to model conclusive solutions that may be understood to adhere to broadly classical principles. The "space" of problems, represented by the dialogue structure between T and T_C, is a space of potential proofs and refutations. The "space" of solutions, represented by the Boolean topos T_B, is a space of actual and conclusive proofs and refutations. In other words, the proof-theoretic construction adequate to co-constructive logic lies beneath what is, essentially, a bivalent structure of conclusive propositions.

7. Backmatter

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